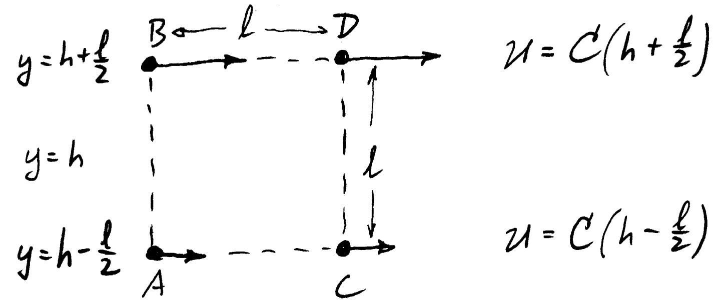
UE Fluids Problem F11-12 Solution Fall '07

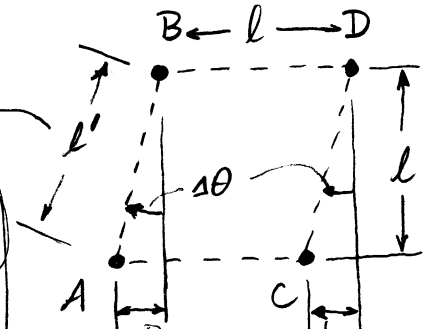
a) This is 2D, so $\vec{\xi} = \xi \hat{k}$, $\boxed{\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - C = -C}$

$\boxed{\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0 + C) = \frac{1}{2}C}$, $\boxed{\varepsilon_{xz} = 0,\ \varepsilon_{yz} = 0}$ (2D)

b) Velocities at the corners, using $u = Cy$, $v = 0$



Relative positions at Δt later time.



$\boxed{\Delta\theta = \arctan\left(\frac{C l \Delta t}{l}\right) = \arctan(C\Delta t)}$

$\boxed{l' = \sqrt{l^2 + (C l \Delta t)^2}}$ or $\boxed{l' = l\sqrt{1 + (C\Delta t)^2}}$

$\Delta x_{DC} = (u_D - u_C)\Delta t = C l \Delta t$

$\Delta x_{BA} = (u_B - u_A)\Delta t = C l \Delta t$

c) Try to find a $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = -v = 0$, $\frac{\partial \psi}{\partial y} = u = Cy$

$\frac{\partial \psi}{\partial x} = 0 \rightarrow$ implies $\psi(x,y) = f(y)$ only (1)

$\frac{\partial \psi}{\partial y} = Cy \rightarrow$ implies $\psi(x,y) = \frac{1}{2}Cy^2 + g(x)$ (2)

For (1) and (2) to be consistent, we must have

$f(y) = \frac{1}{2}Cy^2 + \phi$, $g(x) = \phi$ (same constant ϕ)

$\therefore \boxed{\psi(x,y) = \frac{1}{2}Cy^2 + \cancel{\phi}}$, (can in general set $\phi = 0$)

For this case $\boxed{\xi = -C \neq 0}$, so $\boxed{\phi(x,y) \text{ does not exist.}}$

Part d), repeat a), b), c) for $u = Cy^{1/2}$, $v = 0$

a) $\boxed{\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{1}{2}\frac{C}{y^{1/2}} = -\frac{C}{2y^{1/2}}}$. $\boxed{\text{At } y = h,\ \xi = -\frac{C}{2h^{1/2}}}$

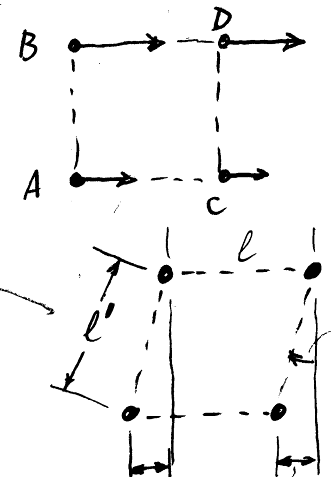
$\boxed{\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{C}{4y^{1/2}}}$. $\boxed{\text{At } y = h,\ \varepsilon_{xy} = \frac{C}{4h^{1/2}}}$

b) At $y = h + l/2$, $u_B = C(h + l/2)^{1/2} = Ch^{1/2}\left(1 + \frac{1}{2}\frac{l}{h}\right)^{1/2}$ note $\frac{1}{2}\frac{l}{h} \ll 1$

Using Taylor series $(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2) \approx 1 + \frac{1}{2}\epsilon$ if $\epsilon \ll 1$

$$u_B = u_D \simeq Ch^{1/2}\left(1 + \frac{1}{4}\frac{l}{h}\right)$$

similarly, $u_A = u_C \approx Ch^{1/2}\left(1 - \frac{1}{4}\frac{l}{h}\right)$



Δt later...

$$\boxed{\begin{aligned} l' &= \sqrt{l^2 + \left(\frac{Cl}{2h^{1/2}}\Delta t\right)^2} \\ l' &= l\sqrt{1 + \frac{C^2 l^2}{4h}\Delta t^2} \end{aligned}}$$

$$\boxed{\Delta\theta = \arctan\left(\frac{C}{2h^{1/2}}\Delta t\right)}$$

$$\Delta X_{BA} = (u_B - u_A)\Delta t = Ch^{1/2}\cdot\frac{1}{2}\frac{l}{h}\Delta t$$

or $\Delta X_{BA} = \frac{Cl}{2h^{1/2}}\Delta t$

Note: Could have also used relations from F11 notes:

$$\Delta X_{BA} \equiv \Delta X_B - \Delta X_A = \left.\frac{\partial u}{\partial y} l \Delta t\right|_{y=h} = \frac{C}{2h^{1/2}} \cdot l\Delta t \quad \text{same result.}$$

c) $\frac{\partial \psi}{\partial x} = -v = 0 \rightarrow \psi(x,y) = f(y)$

$\frac{\partial \psi}{\partial y} = u = Cy^{1/2} \rightarrow \psi(x,y) = \frac{2}{3}Cy^{3/2} + g(x)$

$f(y) = \frac{2}{3}Cy^{3/2} + \phi$

$g(x) = \phi$

$\therefore \boxed{\psi(x,y) = \frac{2}{3}Cy^{3/2} + \phi}$ can set $\phi = 0$

Again in this case $\xi = -\frac{C}{2h^{1/2}} \neq 0$, so $\boxed{\phi(x,y) \text{ does not exist}}$

UE Fluids Problem F13 Solution Fall '07

1) $\dot{m} = \rho V A = \rho_c V_c A_c$

Measured / known quantities: p, T, p_c, also A, A_c

Use additional relations:

2) $p = \rho R T \rightarrow \boxed{\rho = \frac{p}{RT}} \simeq \rho_c$ (low speed flow)

3) $p_0 = p_{0c}$ (Bernoulli)

or $p + \frac{1}{2}\rho V^2 = p_c + \frac{1}{2}\rho V_c^2$

From 1) we have $V_c = V \cdot \frac{\rho A}{\rho_c A_c} \simeq V \frac{A}{A_c}$

so $\frac{1}{2}\rho V^2 - \frac{1}{2}\rho\left(V\frac{A}{A_c}\right)^2 = p_c - p$

or $\frac{1}{2}\rho V^2\left[1 + \left(\frac{A}{A_c}\right)^2\right] = p_c - p$

or $V = \left[\frac{2(p_c - p)}{1 + (A/A_c)^2}\right]^{1/2}$

Finally, $\boxed{\dot{m} = \rho\left[\frac{2(p_c - p)}{1 + (A/A_c)^2}\right]^{1/2} A}$

or equivalent $\boxed{\dot{m} = \rho\left[\frac{2(p_c - p)}{1 + (A_c/A)^2}\right]^{1/2} A_c}$

Or if we plug in explicit ρ expression from above:

$$\dot{m} = \frac{p}{RT}\left[\frac{2(p_c - p)}{1 + (A/A_c)^2}\right]^{1/2} A$$

or $$\dot{m} = \frac{p}{RT}\left[\frac{2(p_c - p)}{1 + (A_c/A)^2}\right]^{1/2} A_c$$